Wave Motion in Hydrodynamics.

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The mathematical subject of Hydrodynamics is still, in some respects, in an elementary stage, insomuch as every fresh problem solved constitutes a distinct advance of the subject.

One of the most important applications of the theory of Hydrodynamics is to the question of the motion of Waves under gravity and other causes, and as the investigations on this subject are for the most part scattered about in various scientific periodicals, I propose in this article to collect together the chief results hitherto obtained, and to give also a general connected account of the mathematical theory, at the same time attempting to develop it in some directions.

In the mathematical treatment of Wave Motion we are constrained at present to employ the approximation of supposing the velocities of the liquid particles due to the wave motion to be sufficiently small for the squares, etc., of the particle velocities to be neglected; although it is singular that this approximation is not required in the first problem of wave motion ever solved, discovered by Gerstner in 1802, and afterwards independently by Rankine in 1862 (Stokes, Mathematical and Physical Papers, I, p. 219).

A list of the principal papers on the subject of Wave Motion and of their authors will be found in the Report on Recent Progress in Hydrodynamics, by W. M. Hicks, F. R. S., presented to the British Association.

1. The most convenient order to employ in the mathematical treatment of a problem in the subject of Wave Motion is: (I) The determination of the velocity function ϕ, or stream function ψ, satisfying the equation of continuity; (II) The determination of the boundary conditions to be satisfied at the sides of the containing vessel; (III) The most difficult part, the determination of the conditions to be satisfied in order that the free surface should be a surface of equal pressure, or, more generally, at the surface of separation of two liquids there should be no discontinuity of pressure.

To secure uniformity of notation in the treatment of waves under gravity, we shall suppose the co-ordinate axis Oz drawn vertically upwards, and the plane xOy taken generally in the undisturbed horizontal plane of the surface of separation of two liquids at which the wave motion is apparent, and then the axis Ox will be taken in the direction of propagation of the waves when straight-crested, and, therefore, perpendicular to the crests.

Then, to determine the wave motion in still water (still except for the slight disturbance of the wave motion), we must determine a velocity function ϕ. (I) Satisfying the equation of continuity for liquids:

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0.$$

(II) Satisfying the condition

$$\frac{\partial\phi}{\partial\nu}=0$$

at a fixed boundary, $\partial\nu$ denoting an element of the outward-drawn normal of the boundary, or, more generally,

$$\frac{\partial\phi}{\partial\nu}=n,$$

the normal component velocity of the boundary, when movable. (III) Satisfying at the surface $z=0$, supposed a surface of equal or of no discontinuity of pressure, the dynamical equation

$$\frac{p}{\rho}+gz+\frac{\partial\phi}{\partial t}=H,$$

a constant, neglecting the squares of the velocities of the liquid particles.

At a free surface p is constant, and, therefore, $\frac{\partial p}{\partial t}=0$, so that, η denoting the elevation of the free surface,

$$g\frac{\partial\eta}{\partial t}+\frac{\partial^2\phi}{\partial t^2}=0.$$

But
$$\frac{\partial\eta}{\partial t}=\frac{\partial\phi}{\partial z},$$

and
$$\frac{\partial^2\phi}{\partial t^2}=-\frac{g}{l}\phi,$$

where l denotes the length of the equivalent simple pendulum of the wave motion, so that

$$l\frac{\partial\phi}{\partial z}=\phi;$$

at the free surface $z=0$, or $z=h$, some constant.

2. When surface tension T is taken into account, this equation must be replaced by
$$l\frac{\partial\phi}{\partial z}+\frac{Tl}{g\rho}\frac{\partial^3\phi}{\partial z^3}=\phi,$$
an equation due to Kolacek (*vide* Fortschritte der Mathematik, 1878). For, if ∂p denotes the excess of pressure in the liquid just below the capillary film over the external pressure above,
$$\frac{\partial p}{\rho}+g\eta+\frac{\partial\phi}{\partial t}=0.$$
But r and r', denoting the radii of curvature of any two vertical sections of the free surfaces by perpendicular planes,
$$-\partial p=T\left(\frac{1}{r}+\frac{1}{r'}\right)=T\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)$$
to one order of approximation, so that
$$T\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)=g\rho\eta+\rho\frac{\partial\phi}{\partial t};$$
and differentiating with respect to t, and replacing $\frac{d\eta}{dt}$ by $\frac{\partial\phi}{\partial z}$, remembering also that
$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-\frac{\partial^2\phi}{\partial z^2},$$
then
$$-T\frac{\partial^3\phi}{\partial z^3}=g\rho\frac{\partial\phi}{\partial z}-\frac{g\rho}{l}\phi,$$
or
$$l\frac{\partial\phi}{\partial z}+\frac{Tl}{g\rho}\frac{\partial^3\phi}{\partial z^3}=\phi.$$

3. *Digression on the Hyperbolic Functions.*

In the course of our investigations we shall require certain functions, called Hyperbolic Functions, from their connection with the hyperbola, which are analogous to the functions of the circle defined in ordinary trigonometry. As these functions are not defined and explained in all the ordinary text-books, we shall, for convenience, proceed to do so as follows:

(I) $\frac{1}{2}(e^v+e^{-v})$ is called the *hyperbolic cosine* of v, and is denoted by $\cosh v$.

(II) $\frac{1}{2}(e^v-e^{-v})$ is called the *hyperbolic sine* of v, and is denoted by $\sinh v$.

(III) $\frac{e^v-e^{-v}}{e^v+e^{-v}}=\frac{\sinh v}{\cosh v}$ is called the *hyperbolic tangent* of v, and is denoted by $\tanh v$; and so on, by analogy, with the rest of the circular functions.

From the *exponential* values of the cosine and sine, viz.,

$$\cos u = \frac{1}{2}(e^{iu} + e^{-iu}), \quad \sin u = \frac{1}{2i}(e^{iu} - e^{-iu}),$$

when i denotes $\sqrt{-1}$, we see, by putting $u = iv$, that $\cos iv = \cosh v$, $\sin iv = i \sinh v$, $\tan iv = i \tanh v$, etc.; also,

$$\cos(u + iv) = \cos u \cosh v - i \sin u \sinh v,$$
$$\sin(u + iv) = \sin u \cosh v + i \cos u \sinh v, \text{ etc.,}$$

formulæ of great use hereafter. Therefore, also,

$$\cosh(u + iv) = \cosh u \cos v - i \sinh u \sin v,$$
$$\sinh(u + iv) = \sinh u \cos v - i \cosh u \sin v.$$

Analogous to the ordinary formulæ of circular trigonometry, we have

$$\cosh(\alpha + \beta) = \cosh\alpha \cosh\beta + \sinh\alpha \sinh\beta,$$
$$\sinh(\alpha + \beta) = \sinh\alpha \cosh\beta + \cosh\alpha \sinh\beta,$$
$$\tanh(\alpha + \beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha \tanh\beta},$$
$$\sinh\gamma + \sinh\delta = 2 \sinh\frac{1}{2}(\gamma + \delta) \cosh\frac{1}{2}(\gamma - \delta),$$
$$\sinh\gamma - \sinh\delta = 2 \cosh\frac{1}{2}(\gamma + \delta) \sinh\frac{1}{2}(\gamma - \delta),$$
$$\cosh\gamma + \cosh\delta = 2 \cosh\frac{1}{2}(\gamma + \delta) \cosh\frac{1}{2}(\gamma - \delta),$$
$$\cosh\gamma - \cosh\delta = 2 \sinh\frac{1}{2}(\gamma + \delta) \sinh\frac{1}{2}(\gamma - \delta),$$

and so on.

4. *Waves in Still Water of Uniform Depth.*

Supposing straight-crested waves of length λ propagated in the direction of the axis of x with velocity U, we may begin by supposing the velocity function

$$\phi = f(z) \cos(mx - nt),$$

where $m = 2\pi/\lambda$, $n = 2\pi U/\lambda$, and $n^2 = g/l$, l denoting the length of the equivalent simple pendulum.

Then, from the equation of continuity,

$$\frac{d^2f}{dz^2} - m^2 f = 0,$$

the solution of which is

$$f(z) = ae^{mz} + be^{-mz};$$

or, using hyperbolic functions,

$$f(z) = P \cosh mz + Q \sinh mz;$$

or, subject to the condition that

$$\frac{\partial \phi}{\partial z} = 0, \text{ when } z = -h,$$

h denoting the depth of the water,

$$f(z) = A \cosh m(z+h);$$

so that

$$\phi = A \cosh m(z+h) \cos(mx - nt).$$

Then, at the free surface $z = 0$,

$$l \frac{\partial \phi}{\partial z} = \phi;$$

or,

$$ml \sinh mh = \cosh mh;$$

or,

$$ml = \coth mh.$$

Then,

$$U^2 = \frac{n^2}{m^2} = \frac{g}{m^2 l}$$

$$= \frac{g}{m} \tanh ml = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda},$$

the well-known expression for the wave velocity.

When h/λ is small, we can replace $\tanh(2\pi h/\lambda)$ by $2\pi h/\lambda$, and then

$$U^2 = gh,$$

Kelland, Scott, Russell and Green's expressions for the wave velocity when the wave length is great compared with the depth of water.

When h/λ is large, we can replace $\tanh(2\pi h/\lambda)$ by unity, and then

$$U^2 = g\lambda/2\pi,$$

agreeing with Gerstner's and Rankine's expressions for the wave velocity in water of great depth.

5. Next, suppose there is a surface tension T at the free surface; then the condition

$$l \frac{\partial \phi}{\partial z} + \frac{Tl}{g\rho} \frac{\partial^3 \phi}{\partial z^3} = \phi,$$

when $z = 0$, leads to the relation

$$ml \sinh mh + \frac{Tl}{g\rho} m^3 \sinh mh = \cosh mh;$$

or,

$$U^2 = \frac{n^2}{m^2}$$

$$= g\left(\frac{1}{m} + \frac{Tm}{g\rho}\right) \tanh mh$$

$$= \left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda}\right) \tanh \frac{2\pi h}{\lambda},$$

the general expression for the wave velocity under gravity and surface tension combined.

When λ is large or T small, we may put

$$U^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda},$$

as above; but when λ is small, the term $g\lambda/2\pi$ is insensible, and we can put

$$U^2 = \frac{2\pi T}{\rho\lambda} \tanh \frac{2\pi h}{\lambda},$$

the velocity of propagation of ripples of wave length λ due to surface tension T.

Supposing the depth of water h sufficiently large for $\tanh(2\pi h/\lambda)$ to be replaced by unity, then

$$U^2 = \frac{g\lambda}{2\pi} + \frac{2\pi T}{\rho\lambda} = 2\sqrt{\frac{gT}{\rho}} + \left(\sqrt{\frac{g\lambda}{2\pi}} - \sqrt{\frac{2\pi T}{\rho\lambda}}\right)^2,$$

so that the minimum value of U is

$$\sqrt{\left(2\sqrt{\frac{gT}{\rho}}\right)},$$

and the $\lambda = 2\pi\sqrt{(T/g\rho)}$.

Sir W. Thomson proposes to distinguish by the name of *ripples* those waves whose length is less than the above critical value of λ (Phil. Mag. (4) xlii).

6. A slight extension of this problem may be made by supposing the capillary film of the surface to be replaced by a flexible cloth of uniform tension T and uniform superficial density σ resting on the surface of the liquid.

Then, assuming for the liquid motion a velocity function

$$\phi = A \cosh m(z+h) \cos(mx - nt),$$

as before, supposing A a small constant factor, and denoting by η the elevation of the surface, then, when $z = 0$,

$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} = mA \sinh mh \cos(mx - nt),$$

and, therefore, $$\eta = -\frac{m}{n} A \sinh mh \sin(mx - nt).$$

Denoting by ∂p the excess of pressure just below the cloth over the atmospheric pressure above, then

$$\frac{\partial p}{\rho} + g\eta + \frac{\partial\phi}{dt} = 0;$$

or, $$\partial p - g\rho \frac{m}{n} nA \sinh mh \sin(mx - nt) + nA \cosh mh \sin(mx - nt) = 0.$$

But the equation of motion of the cloth is

$$\sigma \frac{d^2\eta}{dt^2} = T \frac{d^2\eta}{dx^2} + \partial p;$$

so that, dropping the common factor $A \sin(mx - nt)$,

$$\sigma mn \sinh mh = T \frac{m^3}{n} \sinh mh + g\rho \frac{m}{n} \sinh mh - \rho n \cosh mh;$$

or,

$$U^2 = \frac{n^2}{m^2} = \frac{g\rho/m + Tm}{\rho \coth mh + \sigma m},$$

giving U the velocity of propagation of the waves, and reducing, when $\sigma = 0$, to the preceding case of a capillary film.

7. *Waves in Ice of Uniform Thickness Resting on Water of Uniform Depth.*

If the water is covered with ice, then the equation of vibration of the surface must be replaced by

$$\sigma \frac{d^2\eta}{dt^2} = -L \frac{d^4\eta}{dx^4} + \partial p,$$

where L denotes the flexural rigidity of the ice, the vibrations being now of the nature called *lateral* vibrations (Rayleigh, Theory of Sound, I, Chapter 8, §163), the *inertia* of each vertical section of the ice being supposed concentrated at the centre.

Then if e denotes the thickness of the ice, and E Young's modulus of elasticity,

$$L = \frac{1}{12} e^3 E, \text{ and } \sigma = e\rho,$$

supposing the ice of the same density as water, so that now

$$\sigma mn \sinh mh = -L \frac{m^5}{n} \sinh mh + g\rho \frac{m}{n} \sinh mh - \rho n \cosh mh;$$

or,

$$U^2 = \frac{n^2}{m^2} = \frac{g\rho/m + Lm^3}{\rho \coth mh + \sigma m}$$

$$= \frac{g\lambda/2\pi + \frac{2}{3}\pi^3 e^3 E/\rho\lambda^3}{\coth(2\pi h/\lambda) + 2\pi e/\lambda},$$

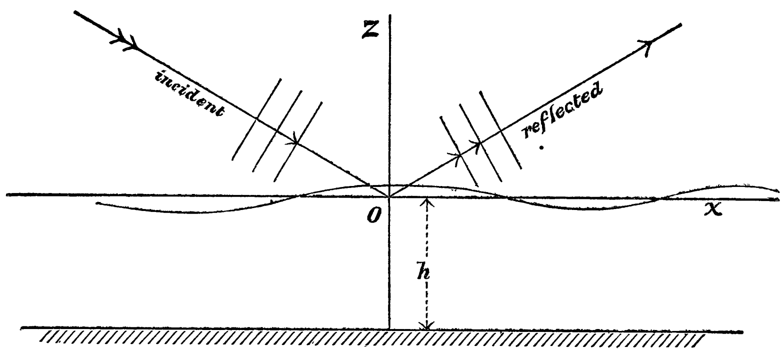
giving the velocity of propagation of waves of length λ in ice of thickness e, resting on water of uniform depth h.

It is remarkable that ice was the first substance for which an experimental determination of E was attempted, as described in Young's Lectures on Natural Philosophy.

8. *Waves in Water of Uniform Depth Established and Maintained by Impinging Waves of Sound in Air.*

Suppose the preceding kind of wave motion in conjunction with plane waves of sound impinging at an angle β, we have thus an illustration of a *forced*, or rather *controlled*, wave motion in the water due to *free* waves in the air. Let
$$\xi = B \sin \{m(x \sin\beta - z \cos\beta) - nt + \alpha\}$$
represent the normal displacement in the incident wave of sound, and
$$\xi_1 = B_1 \sin \{m(x \sin\beta + z \cos\beta) - nt + \alpha_1\},$$
in the reflected wave; and let
$$\eta = b \sin(mx \sin\beta - nt)$$
represent the displacement of the surface of the water. Then, at this surface, we must have
$$\eta = (\xi_1 - \xi) \cos\beta,$$
when $z = 0$, in order that there should be no separation of the air from the water, so that
$$b \sec\beta \sin(mx \sin\beta - nt) = B_1 \sin(mx \sin\beta - nt + \alpha_1) - B \sin(mx \sin\beta - nt + \alpha)$$
for all values of x, leading to the equations
$$\left.\begin{aligned} \beta_1 \cos\alpha_1 - B \cos\alpha &= b \sec\beta \\ \beta_1 \sin\alpha_1 - B \sin\alpha &= 0 \end{aligned}\right\}.$$

The velocity function of the motion in the water must be of the form
$$\phi = A \cosh\{m(z+h)\sin\beta\} \cos(mx \sin\beta - nt);$$



and then, since
$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z},$$
when $z = 0$, therefore,
$$-nb = mA \sin\beta \sinh(mh \sin\beta).$$

From the dynamical equation

$$\frac{p}{\rho} + gz + \frac{\partial \phi}{\partial t} = H$$

we have, denoting by ∂p the periodic part of p just below the surface of the water,

$$\frac{\partial p}{\rho} + g\eta + \frac{\partial \phi}{\partial t} = 0;$$

or,

$$\partial p = -g\rho\eta - \rho\frac{\partial \phi}{\partial t}$$

$$= -\{g\rho b + \rho n A \cosh(mh \sin\beta)\} \sin(mx \sin\beta - nt).$$

Suppose, now, at the surface of separation of the air and the water there is a film or cloth of tension T and superficial density σ; and we might also, if we like, suppose the film to possess flexural rigidity L like ice, without much additional complication; then, at the surface of separation,

$$\sigma\frac{d^2\eta}{dt^2} = T\frac{d^2\eta}{dx^2} - L\frac{d^4\eta}{dx^4} + \partial p - \partial p',$$

when $z = 0$, where $\partial p'$ denotes the periodic part of the pressure in the air due to the wave motion.

Now, ρ' denoting the density of air and a the velocity of the sound waves, we have

$$\frac{n^2}{m^2} = a^2 = \gamma\frac{p'}{\rho'},$$

and the cubical elasticity

$$\rho'\frac{dp'}{d\rho'} = \gamma p';$$

so that

$$\partial p' = \gamma p'\frac{\partial \rho'}{\rho'} = -\gamma p' s = -a^2\rho' s,$$

s denoting the cubical expansion; so that

$$s = \frac{\partial}{\partial x}(\xi_1 + \xi/\sin\beta) + \frac{\partial}{\partial z}(\xi_1 - \xi)\cos\beta$$

$$= mB_1 \cos\{m(x\sin\beta + z\cos\beta) - nt + \alpha_1\}$$

$$+ mB\cos\{m(x\sin\beta - z\cos\beta) - nt + \alpha\}.$$

Therefore, when $z = 0$,

$$\partial p' = -a^2\rho' m\{B_1\cos(mx\sin\beta - nt + \alpha_1) + B\cos(mx\sin\beta - nt + \alpha)\}$$

$$= a^2\rho' m(B_1\sin\alpha_1 + B\sin\alpha)\sin(mx\sin\beta - nt),$$

provided that

$$B_1\cos\alpha_1 + B\cos\alpha = 0,$$

which, combined with the previous equations, gives

$$\frac{B_1}{B} = \frac{\sin\alpha}{\sin\alpha_1} = -\frac{\cos\alpha}{\cos\alpha_1};$$

or, $$\tan\alpha = -\tan\alpha_1,\ \alpha = -\alpha_1,$$

and $$B_1 = -B = \frac{1}{2} b \sec\alpha \sec\beta.$$

Then, since $$b = -\frac{m}{n} A \sin\beta \sinh(mh\sin\beta),$$

the boundary condition becomes, when the common factor $A\sin(mx\sin\beta - nt)$ is dropped,

$$\sigma mn \sin\beta \sinh(mh\sin\beta) = T\frac{m^3}{n}\sin^3\beta \sinh(mh\sin\beta)$$
$$-L\frac{m^5}{n}\sin^5\beta\sinh(mh\sin\beta) + g\rho\frac{m}{n}\sin\beta\sinh(mh\sin\beta)$$
$$-\rho n\cosh(mh\sin\beta) - a^2\rho'\frac{m^2}{n}\tan\alpha\tan\beta\sinh(mh\sin\beta);$$

or, $$\rho' n\tan\alpha\tan\beta = -\sigma mn\sin\beta + T\frac{m^3}{n}\sin^3\beta$$
$$-L\frac{m^5}{n}\sin^5\beta + g\rho\frac{m}{n}\sin\beta - \rho n\coth(mh\sin\beta),$$

giving $\tan\alpha$, and therefore α, the change of phase of the sound wave in being reflected at the surface of the water.

9. *Reflection and Refraction of Plane Waves of Sound by a Plane Curtain.*

Suppose, now, that the cloth, instead of resting on the surface of a liquid, is the plane surface of separation of two elastic fluids of different densities ρ and ρ', but necessarily of the same pressure p when at rest, the cloth being now supposed vertical to abstract the curving effect of gravity; let us now investigate the reflection and refraction of plane waves of sound, impinging and being reflected at an angle β in the first medium of density ρ, and being refracted at an angle β' in the second medium of density ρ'.

Denoting as before by

$$\xi = B\sin\{m(x\sin\beta - z\cos\beta) - nt + \alpha\}$$

the normal displacement of the incident waves, and by

$$\xi_1 = B_1\sin\{m(x\sin\beta + z\cos\beta) - nt + \alpha_1\}$$

of the reflected, and by

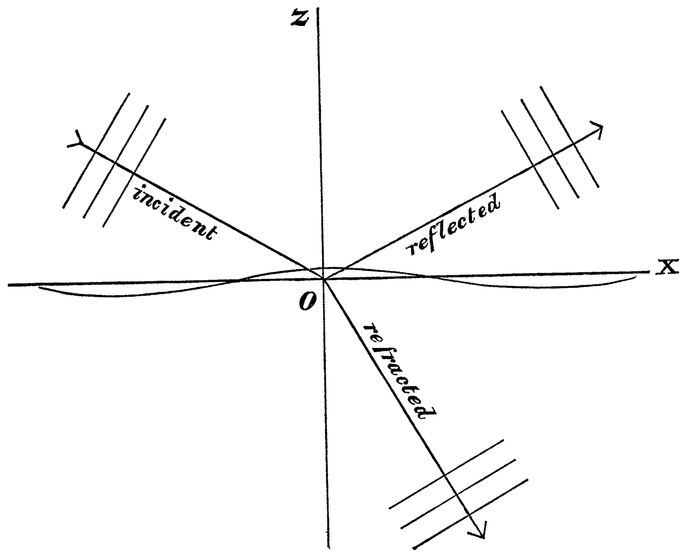
$$\xi' = B'\sin\{m'(x\sin\beta' - z\cos\beta') - nt + \alpha'\}$$

of the refracted wave; then

(I) $$m\sin\beta = m'\sin\beta',$$ the *Law of Refraction.*

Also, denoting the displacement of the surface of separation, the cloth or curtain, by $\eta = b \sin(mx \sin\beta - nt)$;
then, as before, when $z = 0$,

$$\eta = (\xi_1 - \xi)\cos\beta = -\xi' \cos\beta',$$



leading to

$$\begin{aligned} B_1 \cos\alpha_1 - B\cos\alpha &= b\sec\beta, \\ B_1 \sin\alpha_1 - B\sin\alpha &= 0, \\ B'\cos\alpha' &= -b\sec\beta', \\ B'\sin\alpha' &= 0; \end{aligned}$$

so that $\alpha' = 0$, and $B' = -b\sec\beta'$.

For the motion of the curtain,

$$\sigma\frac{d^2\eta}{dt^2} = T\frac{d^2\eta}{dx^2} + \partial p' - \partial p,$$

where $\partial p = -a^2\rho s, \quad \partial p' = -a'^2\rho' s'$,
s and s' denoting the cubical expansion in the two media in the neighborhood of $z = 0$; also, a and a' denoting the velocity of sound in the two media. Then

$$s = mB_1\cos(mx\sin\beta - nt + \alpha_1) + mB\cos(mx - nt + \alpha),$$
$$s' = m'B'\cos(mx\sin\beta - nt);$$

so that

$$-\sigma\frac{d^2\eta}{dt^2} + T\frac{d^2\eta}{dx^2} = (\sigma n^2 - Tm^2\sin^2\beta)\, b\sin(mx\sin\beta - nt)$$
$$= \partial p - \partial p' = -a^2\rho m\{B_1\cos(mx\sin\beta - nt + \alpha_1) + B\cos(mx - nt + \alpha)\}$$
$$+ a'^2\rho' m' B'\cos(mx\sin\beta - nt),$$

for all values of x and t, leading to the conditions

$$(\sigma n^2 - Tm^2 \sin^2 \beta)\, b = a^2 \rho m\, (B_1 \sin \alpha_1 + B \sin \alpha)$$

and

$$0 = - a^2 \rho m\, (B_1 \cos \alpha_1 + B \cos \alpha) + a'^2 \rho' m' B'.$$

Therefore, since $a^2 = n^2/m^2$, $a'^2 = n^2/m'^2$,

$$B_1 \sin \alpha_1 = B \sin \alpha = \frac{\sigma n^2 - Tm^2 \sin^2 \beta}{2 \rho n^2}\, mb = \frac{\sigma n^2 - Tm^2 \sin^2 \beta}{2a^2 \rho m}\, b,$$

and

$$B_1 \cos \alpha_1 + B \cos \alpha = \frac{a'^2 \rho' m'}{a^2 \rho m}\, B',$$

$$= - \frac{a'^2 \rho' m'}{a^2 \rho m}\, b \sec \beta';$$

also, $B_1 \cos \alpha_1 - B \cos \alpha = b \sec \beta$;

so that

$$B_1 \cos \alpha_1 = \frac{a^2 \rho m \sec \beta - a'^2 \rho' m' \sec \beta'}{2a^2 \rho m}\, b,$$

$$B \cos \alpha = - \frac{a^2 \rho m \sec \beta + a'^2 \rho' m' \sec \beta'}{2a^2 \rho m}\, b,$$

giving

$$\cot \alpha = - \frac{a^2 \rho m \sec \beta + a'^2 \rho' m' \sec \beta'}{\sigma n^2 - Tm^2 \sin^2 \beta},$$

$$\cot \alpha_1 = \frac{a^2 \rho m \sec \beta - a'^2 \rho' m' \sec \beta'}{\sigma n^2 - Tm^2 \sin^2 \beta},$$

whence the change of phase by reflection is determined; also,

$$B^2 : B_1^2 : B'^2$$

$$= (\sigma n^2 - Tm^2 \sin^2 \beta)^2 + (a^2 \rho m \sec \beta + a'^2 \rho' m' \sec \beta')^2,$$

$$: (\sigma n^2 - Tm^2 \sin^2 \beta) + (a^2 \rho m \sec \beta - a'^2 \rho' m' \sec \beta')^2,$$

$$: \quad 4a^4 \rho^2 m^2 \sec^2 \beta',$$

giving the ratios of the intensities of the incident, reflected and refracted waves.

Put $\sigma = 0$ and $T = 0$ and we obtain the results of the cases considered by Green in his paper on the Reflection and Refraction of Sound, published in the Transactions of the Cambridge Philosophical Society, 1838, and republished by Ferrers in the Mathematical Papers of the late George Green, 1871.

Then

$$\alpha = 0, \ \alpha_1 = 0,$$

and

$$B_1 = \frac{a \rho \sec \beta - a' \rho' \sec \beta'}{2a\rho}\, b,$$

$$B = - \frac{a \rho \sec \beta + a' \rho' \sec \beta'}{2a\rho}\, b,$$

since

$$am = a'm' = n.$$

We might, as in Green's paper, have supposed the incident, reflected and refracted plane waves given by

$$\phi = f\{m(x\sin\beta - z\cos\beta) - nt + \alpha\},$$
$$\phi_1 = F\{m(x\sin\beta + z\cos\beta) - nt + \alpha_1\},$$
$$\phi' = f_1\{m'(x\sin\beta' - z\cos\beta') - nt + \alpha'\},$$

and the displacement of the curtain by

$$\eta = f_2\{mx\sin\beta - nt\},$$

when f, F, f_1, f_2 denote arbitrary functions, and determine the conditions to be satisfied as before.

10. *Theory of Long Waves in Canals.*

In this theory the vertical motion of the liquid particles is supposed insensible compared with the horizontal motion, and the depth of water small compared with the wave length; so that

$$U^2 = gh,$$

as before in § 4, for water of uniform depth.

This is proved independently by supposing the pressure at any depth the same as the hydrostatic pressure due to the depth below the free surface; so that ξ, denoting the horizontal displacement of a liquid particle, and η the elevation of the free surface, then

$$\rho\frac{d^2\xi}{dt^2} = -\frac{dp}{dx}$$

and

$$\frac{dp}{dx} = g\rho\frac{d\eta}{dx};$$

so that

$$\frac{d^2\xi}{dt^2} = -g\frac{d\eta}{dx}.$$

But the equation of continuity leads to the condition

$$b(h+\eta)\left(1+\frac{d\xi}{dx}\right) = bh;$$

or,

$$\eta + h\frac{d\xi}{dx} = 0,$$

to one order of approximation, b denoting the breadth and h the depth of the canal of water. Then

$$\frac{d^2\xi}{dt^2} = gh\frac{d^2\xi}{dx^2};$$

so that

$$U = gh,$$

U denoting the velocity of wave propagation.

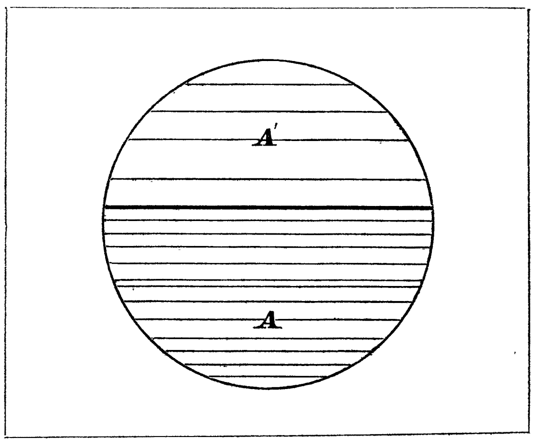
We may, however, generalize this kind of motion, as was done by Kelland, by supposing the cross-section of the canal of any form, but uniform; and then Kelland found

$$U^2 = gA/b,$$

where A is the area of the cross-section, and b, as before, the breadth at the surface of the water.

For, in the more general case, where we consider the waves at the surface of separation of two liquids of densities ρ and ρ' filling a closed uniform horizontal pipe or conduit, so that A and A' denote the cross-sections of the pipe occupied by the liquids, and b the breadth of the plane of separation, we shall find

$$U^2 = \frac{g}{b} \frac{\rho - \rho'}{\dfrac{\rho}{A} + \dfrac{\rho'}{A'}},$$



reducing to gA/b, where $\rho' = 0$.

The simplest way to prove this is to suppose the motion made steady by applying the reversed velocity $-U$, equivalent to considering the motion relative to an origin moving with velocity U.

Then η, denoting the elevation of the surface, and u, u' the small additional velocities in the liquid due to the wave motion,

$$(A + b\eta)(U + u) = AU,$$
$$(A' - b\eta)(U + u') = A'U;$$

or,

$$Au + bU\eta = 0,$$
$$A'u' - bU\eta = 0.$$

Then, ∂p and $\partial p'$, denoting the increments of pressure caused by the wave motion in the liquids just below and just above the surface of separation,

$$\partial p + g\rho\eta + \frac{1}{2}\rho(U+u)^2 - \frac{1}{2}\rho U^2 = 0,$$

$$\partial p' + g\rho'\eta' + \frac{1}{2}\rho'(U+u')^2 - \frac{1}{2}\rho' U^2 = 0;$$

or,

$$\partial p + g\rho\eta + \rho U u = 0,$$

$$\partial p' + g\rho'\eta + \rho' U u' = 0.$$

When there is no capillarity, etc., $\partial p = \partial p'$, so that

$$g(\rho - \rho')\eta = (\rho' u' - \rho u)U$$

$$= \left(\frac{\rho'}{A'} + \frac{\rho}{A}\right) b U^2 \eta;$$

or,

$$U^2 = \frac{g}{b}\,\frac{\rho - \rho'}{\dfrac{\rho}{A} + \dfrac{\rho'}{A'}}.$$

But, with a separating film of tension T,

$$T\frac{d^2\eta}{dx^2} + \partial p - \partial p' = 0;$$

so, if we assume that $\eta = a \cos mx$,

we have

$$T\frac{d^2\eta}{dx^2} = -Tm^2\eta$$

$$= \partial p' - \partial p = g(\rho - \rho')\eta + (\rho u - \rho' u')U$$

$$= g(\rho - \rho')\eta - \left(\frac{\rho}{A} + \frac{\rho'}{A'}\right) b U^2 \eta;$$

or,

$$Tm^2 = \left(\frac{\rho}{A} + \frac{\rho'}{A'}\right) b U^2 - g(\rho + \rho').$$

When the liquids are bounded below and above by horizontal planes, at distances h and h' from the mean plane of separation, this equation becomes

$$Tm^2 = \left(\frac{\rho}{h} + \frac{\rho'}{h'}\right) U^2 - g(\rho - \rho'),$$

an equation which will be found useful as a preliminary to the consideration of the Instability of Jets and its application to the flapping of sails and flags, investigated by Lord Rayleigh (Proceedings of the London Mathematical Society, Vol. X, No. 141).

If the upper liquid had been moving with mean velocity U' different to U, the preceding equations would be replaced by

$$Au + bU\eta = 0,$$

$$A'u' - bU'\eta = 0,$$

$$\partial p + g\rho\eta + \rho U u = 0,$$

$$\partial p' + g\rho\eta + \rho' U' u' = 0,$$

and then

$$Tm^2 = \left(\rho\frac{U^2}{A} + \rho'\frac{U'^2}{A'}\right) b - g(\rho - \rho').$$

11. *Waves at the Surface of Separation of Two Liquids.*

The preceding case suggests the consideration of the general case of waves at the surface of separation of two liquids of different densities, and consequently a horizontal plane, when the liquids are either still except for the wave motion, or are flowing across each other with given mean uniform velocities, in which case the liquids must be bounded above and below by horizontal plane barriers if these velocities are not in the same direction.

First, when the liquids are still, we must have

$$\phi = A \cosh m(z+h) \cos(mx - nt)$$

in the lower liquid, of depth h, as before, and

$$\phi' = A' \cosh m(z-h') \cos(mx-nt)$$

in the upper liquid, of depth h', suppose; and then, if

$$\eta = a \sin(mx-nt)$$

represents the displacement of the surface of separation, we must have

$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} = \frac{\partial\phi'}{\partial z},$$

when $z=0$, in order that there should be no separation of the liquids; consequently,

$$-na = mA \sinh mh = -mA' \sinh mh'.$$

Again, from the hydrodynamical equation

$$\frac{p}{\rho} + gz + \frac{d\phi}{dt} = H$$

we obtain, at the surface of separation,

$$\partial p + g\rho\eta + \rho\frac{d\phi}{dt} = 0$$

in the lower liquid, just below the surface of separation, and

$$\partial p' + g\rho'\eta + \rho'\frac{d\phi'}{dt} = 0$$

in the upper liquid, just above.

Neglecting capillarity, etc., $\partial p = \partial p'$, and, therefore,

$$g\rho\eta + \rho\frac{d\phi}{dt} = g\rho'\eta + \rho'\frac{d\phi'}{dt};$$

or,

$$g(\rho-\rho')\eta = -\rho\frac{d\phi}{dt} + \rho'\frac{d\phi'}{dt}$$

$$= (-\rho nA \cosh mh + \rho' nA' \cosh mh') \cos(mx-nt)$$

$$= \left(\rho\frac{n^2}{m}\coth mh + \rho'\frac{n^2}{m}\coth mh'\right)\eta;$$

so that

$$U^2 = \frac{n^2}{m^2} = \frac{g}{m}\frac{\rho-\rho'}{\rho \coth mh + \rho' \coth mh'}.$$

(Stokes, On the Theory of Oscillatory Waves, Cam. Phil. Trans., Vol. VIII, p. 441; republished in Mathematical and Physical Papers, Vol. I, p. 212.)

Putting $\rho' = 0$, we obtain

$$U^2 = \frac{g}{m} \tanh mh = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda},$$

as before.

When λ is small compared with h and h', then mh and mh' are large, and we may replace coth mh and coth mh' by unity, and then

$$U^2 = \frac{g}{m} \frac{\rho - \rho'}{\rho + \rho'},$$

which, when the ratio ρ'/ρ is small, as is the case of air on water, can be replaced by

$$U^2 = \frac{g}{m}\left(1 - 2\frac{\rho'}{\rho}\right).$$

12. Suppose, now, the upper liquid is moving, like the wind, over the surface of the lower liquid with velocity V', and we wish to determine U, the velocity of propagation of waves of length λ at the surface of separation.

For generality, we shall suppose the lower liquid also moving with velocity V, and seek to determine the new relation connecting V, V' and U.

The simplest way is to take a moving origin or plane of yz, moving with velocity V in the direction of the axis of x, the direction of wave propagation, and to consider the relative motion of the liquid, which will now be steady relatively to the moving co-ordinate axes.

This is equivalent to supposing the motion made steady by impressing the reversed velocity $-U$ on the system. Then we must put

$$\phi = (V - U)x + A \cosh m(z + h) \cos mx,$$
$$\phi' = (V' - U)x + A' \cosh m(z - h') \cos mx;$$

so that, now introducing the conjugate current functions ψ and ψ',

$$\psi = (V - U)z - A \sinh m(z - h) \sin mx,$$
$$\psi' = (V' - U)z - A' \sinh m(z - h') \sin mx.$$

For the liquids not to separate, we must have, when $z = 0$, $\psi = \psi'$; so that

$$A \sinh mh = - A' \sinh mh',$$

the condition obtained otherwise by putting, when $z = 0$,

$$\frac{\partial \phi}{\partial z} = \frac{\partial \psi'}{\partial z}.$$

Now, supposing that the displacement of the surface of separation is given by

$$\eta = a \sin mx,$$

we must have, when $z = 0$,

$$\psi = (V - U)(\eta - a \sin mx),$$
$$\psi' = (V' - U)(\eta - a \sin mx);$$

so that

$$A \sinh mh = (V - U)\, a,$$
$$A' \sinh mh' = -(V' - U)\, a.$$

Also, from the hydrodynamical equations, with the same notation as before,

$$\partial p + g\rho\eta + \frac{1}{2}\rho\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{1}{2}\rho(V - U)^2 = 0,$$
$$\partial p' + g\rho'\eta + \frac{1}{2}\rho'\left(\frac{\partial\phi'}{\partial x}\right)^2 - \frac{1}{2}\rho'(V' - U)^2 = 0;$$

or,

$$\partial p + g\rho\eta - \rho m A (V - U) \cosh mh \sin mx = 0,$$
$$\partial p' + g\rho'\eta - \rho' m A' (V' - U) \cosh mh' \sin mx = 0;$$

or,

$$\partial p = -g\rho\eta + \rho m (V - U)^2 \coth mh\, \eta,$$
$$\partial p' = -g\rho'\eta - \rho' m (V' - U)^2 \coth mh'\, \eta.$$

If there is no capillarity, etc., at the surface of separation, $\partial p = \partial p'$; so that

$$g(\rho - \rho') = \rho m (V - U)^2 \coth mh + \rho' m (V' - U)^2 \coth mh',$$

whence U is determined, when V, V', ρ, ρ' and h, h' are given.

We have supposed here that the current velocities V and V' are in the same direction as U, the wave velocity; but if V and V' make angles α and α' with U, then, in the above expression, V and V' must be replaced by $V \cos \alpha$ and $V' \cos \alpha'$, the components $V \sin \alpha$ and $V' \sin \alpha'$ of the currents perpendicular to the direction of propagation of the waves having no effect upon the determination of U (Encyclopædia Britannica, 9th edition, article Hydromechanics).

13. In the most general case, where the surface of separation is endowed with tension T, superficial density σ and flexural rigidity L, the condition to be satisfied at this surface is

$$\sigma \frac{d^2\eta}{dt^2} = T\frac{d^2\eta}{dx^2} - L\frac{d^4\eta}{dx^4} + \partial p - \partial p'.$$

Now, if

$$\eta = a \sin mx,$$
$$\frac{d^2\eta}{dt^2} = V^2 \frac{d^2\eta}{dx^2} = -n^2\eta,$$
$$\frac{d^2\eta}{dx^2} = -m^2\eta, \quad \frac{d^4\eta}{dx^4} = m^4\eta;$$

also,

$$\partial p - \partial p' = -g(\rho - \rho')\eta + \{\rho m (V - U)^2 \coth mh + \rho' m (V' - U)^2 \coth mh'\}\eta;$$

so that, omitting the common factor η,

$$\sigma n^2 - Tm^2 - Lm^4 - g(\rho - \rho') + \rho m (V - U)^2 \coth mh + \rho' m (V' - U)^2 \coth mh' = 0.$$

The application of this equation to the discussion of the Instability of Jets, including the flapping of flags and sails, has been considered by Rayleigh, as mentioned above.

In the application to a flag we may put $\rho = \rho'$, and replace coth mh and coth mh' by unity; also, $L = 0$, and we may also suppose $T = 0$; then

$$\sigma n^2 + 2\rho m (V - U)^2 = 0,$$

indicating the instability of the motion, and showing that it cannot be represented by a periodic term of small displacement; we must therefore replace in the motion cos $(mx - nt)$ by cosh $(mx - nt)$, sinh $(mx - nt)$, or

$$(P \cosh mx + Q \sinh mx) \cos nt.$$

In the above general equation put $\sigma = 0$, $L = 0$, $V = 0$, and replace coth mh and coth mh' by unity; then

$$Tm^2 + g(\rho - \rho') = \rho m U^2 + \rho' m (V' - U)^2,$$

the equation considered by Thomson for the determination of the ripples produced by wind V' over the surface of still water.

If W is the velocity of ripples of the same wave length without wind,

$$Tm^2 + g(\rho - \rho') = (\rho + \rho') m W^2;$$

or,

$$W^2 = \frac{g}{m}\frac{\rho - \rho'}{\rho + \rho'} + \frac{mT}{\rho + \rho'},$$

the minimum value of which, for different values of m, is

$$W^2 = 2\sqrt{(gT)}\frac{\sqrt{(\rho - \rho')}}{\rho + \rho'},$$

and then $m^2 = g(\rho - \rho')/T$.

But

$$\rho U^2 + \rho'(V' - U)^2 = (\rho + \rho') W^2;$$

so that

$$U = \frac{\rho' V'}{\rho + \rho'} \pm \sqrt{\left\{W^2 - \frac{\rho\rho' V'^2}{(\rho + \rho')^2}\right\}},$$

giving the velocities of the ripples with and against the wind V.

The least value of V'^2 is less than $(\rho + \rho')^2/\rho\rho'$ times the least value of W^2, and, therefore, the least value of V'^2 to produce ripples is

$$2\sqrt{(gT)}\frac{\rho + \rho'}{\rho\rho'}\sqrt{(\rho - \rho')}.$$

If the wind is blowing with velocity greater than this minimum value of V', the plane surface of the water becomes unstable, and ripples are produced (Sir W. Thomson, Phil. Mag., 1871).

14. *Waves in Water Flowing with Variable Velocity fz, Some Function of the Depth z.*

In this manner we may attempt an investigation of the standing waves seen in a sloping current of water, where the velocity varies with the depth in consequence of viscosity and the fluid friction against the bottom; the method, however, will not be very rigorous, as we must begin by assuming fluid friction to account for the varying velocities at different depths, and afterwards neglect fluid friction when we come to consider the superposed wave motion.

Supposing, however, the motion is steady, we must put the current function

$$\psi = Fz - A \sinh m(z+h) \sin mx,$$

so that the mean value of u or $\frac{\partial\psi}{\partial z}$ is $F'z$ or fz, denoting fz by $F'z$. Then, at the bottom of the water,

$$\psi = Fh, \text{ a constant},$$

and at the surface

$$\psi = F0 + \eta f0 - A \sinh mh \sin mx;$$

so that if, at the surface,

$$\eta = a \sin mx,$$

then

$$A \sinh mh = af0.$$

At the surface

$$\partial p + g\rho\eta + \frac{1}{2}\rho\left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{1}{2}\rho(f0)^2 = 0;$$

or,

$$\partial p + g\rho\eta + \rho(\eta f'0 - mA \cosh mh \sin mx) f0 = 0;$$

or,

$$\partial p + \rho\eta\{g + f0f'0 - m(f0)^2 \coth mh\} = 0;$$

so that, if $\partial p = 0$,

$$g + f0f'0 - m(f0)^2 \coth mh = 0.$$

Here $f0$ denotes the velocity of the current at the surface, and $f'0$ the vertical rate of change of the velocity at the surface.

For instance, if the current flows uniformly with velocity V,

$$g - mV^2 \coth mh = 0;$$

or,

$$V^2 = \frac{g}{m} \tanh mh,$$

as before.

For a viscous liquid, flowing over a flat bottom, $f'''z = 0$, $f''z = -\frac{gi}{\mu}$, μ denoting the viscosity and i the slope of the stream, supposed small,

$$f'z = -\frac{gi}{\mu}z + C,$$

$$fz = -\frac{1}{2}\frac{gi}{\mu}z^2 + Cz + V,$$

supposing V the current velocity at the surface (Ency. Brit., Hydraulics). Therefore,

$$g + VC - mV^2 \coth mh = 0;$$

C is generally determined from the condition that the liquid adheres to the bottom, and, therefore, $fh = 0$, giving

$$C = \frac{1}{2}\frac{gi}{\mu}h - \frac{V}{h}.$$

15. In the experimental verification of the above theory of the motion of waves at the surface of separation of two liquids, we can make the wave velocity U as small as we please by making ρ and ρ' nearly equal.

Again, in order to study experimentally the waves in water of uniform depth, the best plan to obtain uniformity of depth is to pour water on the top of mercury (Stokes, Math. and Phys. Papers, I, p. 199). But in this case the mercury forming the bottom of the water will not be fixed, but will itself be set into wave motion, and the modification thus introduced is considered by Stokes on p. 217. This is a particular case of the general conditions to be satisfied when waves are propagated at the surfaces of separation of a number of superincumbent liquids forming horizontal strata, and limited above and below by fixed horizontal planes. If the upper surface is free, the density of the highest stratum of liquid must be supposed zero.

This general theorem has been worked out by Mr. R. R. Webb, and we shall proceed to investigate his results, which were given in the Math. Tripos Examination at Cambridge in Jan., 1884, as follows:

A rectangular pipe whose faces are horizontal and vertical planes is completely filled with $n+1$ liquids; show that the velocities v of propagation of waves of length λ at the surfaces of separation of the strata are given by the equation

$$\begin{vmatrix} A_1, & -B_2, & 0, & 0, & 0, & 0 & . & . & . & . \\ -B_2, & A_2, & -B_3, & 0, & 0, & 0 & . & . & . & . \\ 0, & -B_3, & A_3, & -B_4, & 0, & 0 & . & . & . & . \\ 0, & 0, & -B_4, & A_4, & -B_5, & 0 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0, & -B_{n-1}, & A_{n-1}, & -B_n \\ . & . & . & . & . & . & 0, & 0, & -B_n, & A_n \end{vmatrix} = 0,$$

where $A_r = 2\pi v^2/\lambda\,(\rho_{r+1}.\ \coth 2\pi h_{r+1}/\lambda + \rho_r \coth 2\pi h_r/\lambda) - g\,(\rho_{r+1} - \rho_r)$,
$B_r = 2\pi v^2 \rho_r/\lambda \operatorname{cosech} 2\pi h_r/\lambda$,
and h_r is the equilibrium thickness of the r^{th} stratum of density ρ_r.

In particular, if $\rho_r = r\sigma$, $h_r = ra$, then the $2n$ values of v are included in the formula
$$v = \pm \frac{1}{2}\sqrt{(ga)} \sec\left(\frac{r}{n+1}\frac{\pi}{2}\right),$$
where r is supposed to assume the values $1, 2, 3, \ldots . n$, and λ, the wave length, is supposed very large compared with na.

In order that the velocity function ϕ_r in the r^{th} stratum should satisfy the equation of continuity and the conditions that

$\dfrac{\partial\phi_r}{\partial z} = \dfrac{\partial\phi_{r-1}}{\partial z}$ at the surface of separation of the r^{th} and $r-1^{\text{th}}$ liquids,

$\dfrac{\partial\phi_r}{\partial z} = \dfrac{\partial\phi_{r+1}}{\partial z}$ " " " r^{th} " $r+1^{\text{th}}$ "

then, taking the plane of xy in the upper surface of the pipe, we must have
$$\phi_r = \{C_{r-1}\cosh m\,(z + h_1 + h_2 + \ldots . + h_r)$$
$$- C_r \cosh m\,(z + h_1 + h_2 + \ldots . + h_{r-1})\}\,\frac{\cos(mx - nt)}{m \sinh mh_r},$$
and then
$$\frac{\partial\phi_r}{\partial z} = C_{r-1}\cos(mx - nt),\ \text{when } z = -h_1 - h_2 - \ldots . - h_{r-1},$$
$$\frac{\partial\phi_r}{\partial z} = C_r \cos(mx - nt),\ \text{when } z = -h_1 - h_2 - \ldots . - h_r;$$
and, therefore, if η_r denotes the elevation of the surface of separation between the r^{th} and $r+1^{\text{th}}$ liquids,
$$\frac{d\eta_r}{dt} = C_r \cos(mx - nt),$$
and
$$\eta_r = -\frac{C_r}{n}\sin(mx - nt).$$

To express the fact that there is no discontinuity at this surface of separation, we have the equations
$$\frac{\partial p_r}{\rho_r} + g\eta_r + \frac{\partial\phi_r}{\partial t} = 0,$$
$$\frac{\partial p_{r+1}}{\rho_{r+1}} + g\eta_r + \frac{\partial\phi_{r+1}}{\partial t} = 0;$$
so that, since $\partial p_r = \partial p_{r+1}$,
$$g\rho_r\eta_r + \rho_r\frac{\partial\phi_r}{\partial t} = g\rho_{r+1}\eta_r + \rho_{r+1}\frac{\partial\phi_{r+1}}{\partial t};$$
or,
$$g\,(\rho_{r+1} - \rho_r)\,\eta_r + \rho_{r+1}\frac{\partial\phi_{r+1}}{\partial t} - \rho_r\frac{\partial\phi_r}{\partial t} = 0.$$

Now, when $z = -h_1 - h_2 - \ldots\ldots - h_r$,

$$\frac{\partial\phi_r}{\partial t} = \frac{n}{m}(C_{r-1}\operatorname{cosech} mh_r - C_r \coth mh_r)\sin(mx - nt),$$

$$\frac{\partial\phi_{r+1}}{\partial t} = \frac{n}{m}(C_r \coth mh_{r+1} - C_{r+1}\operatorname{cosech} mh_{r+1})\sin(mx - nt);$$

so that, dropping the common factor $\frac{1}{n}\sin(mx - nt)$,

$$-g(\rho_{r+1} - \rho_r)C_r + \frac{n^2}{m}(C_r\rho_{r+1}\coth mh_{r+1} - C_{r+1}\rho_{r+1}\operatorname{cosech} mh_{r+1}$$
$$- C_{r-1}\rho_r\operatorname{cosech} mh_r + C_r\rho_r\coth mh_r) = 0;$$

or,
$$-C_{r-1}B_r + C_rA_r - C_{r+1}B_{r+1} = 0,$$

since
$$n^2/m = 2\pi v^2/\lambda,\ \ m = 2\pi/\lambda.$$

Also, the top and bottom of the pipe being fixed horizontal planes, $C_0 = 0$ and $C_{n+1} = 0$, so that the elimination of the C's leads to the determinant given above.

When the pipe is open at the top, we can represent the motion by supposing $\rho_1 = 0$, or $B_1 = 0$, and then the particular case for waves when there are two superincumbent liquids with a free surface has been given by Stokes (Math. and Phys. Papers, I, p. 217).

It will be noticed that, although there is no discontinuity in the value of $\frac{\partial\phi}{\partial z}$ at a surface of separation of two strata, there is discontinuity in $\frac{\partial\phi}{\partial x}$, denoting a slipping of one surface over the other, the slipping velocity at the r^{th} surface, where $z = -h_1 - h_2 - \ldots\ldots - h_r$, being

$$\frac{\partial\phi_{r+1}}{\partial x} - \frac{\partial\phi_r}{\partial x}$$
$$= \{C_{r-1}\operatorname{cosech} mh_r - C_r(\coth mh_{r+1} + \coth mh_r) + C_{r+1}\operatorname{cosech} mh_{r+1}\}\sin(mx - nt).$$

This slipping proves a difficulty in the attempt of proceeding from the above investigation of waves in strata of finite thickness to the case of waves in a liquid of variable density arranged in horizontal strata.

When λ is large and m is consequently small, we may replace $\rho_r \coth mh_r$ and $\rho_r \operatorname{cosech} mh_r$ by ρ_r/mh_r; and then, if $\rho_r = r\sigma$, $h_r = ra$,

$$A_r = 2\sigma\frac{n^2}{m^2} - g\sigma a = 2\sigma v^2 - g\sigma a,$$

$$B_r = \sigma\frac{n^2}{m^2} = \sigma v^2;$$

so that the above determinant becomes

$$\left|\begin{array}{ccccccc} C, & 1, & 0, & 0, & 0 & . & . \\ 1, & C, & 1, & 0, & 0 & . & . \\ 0, & 1, & C, & 0, & 0 & . & . \\ 0, & 0, & 1, & C, & 1 & . & . \\ . & . & . & . & . & . & . \end{array}\right| n \text{ rows}$$

$$= 0 \text{, where } C = 2 - \frac{ga}{v^2},$$

the determinant considered by Rayleigh (Theory of Sound, I, p. 131).

It is there proved that, putting $C = 2 \cos \vartheta$, so that

$$v^2 = \frac{1}{4} ga \sec^2 \frac{1}{2} \vartheta,$$

then
$$\vartheta = \frac{r\pi}{n+1},$$

where
$$r = 1, 2, 3, \ldots . n.$$

16. *Waves in Canals with Sloping Sides, or against a Sloping Beach.*

So far, the wave motion considered has only involved two co-ordinates, x and z, and might be considered limited by any two fixed vertical planes perpendicular to the axis of y.

In the case of the canal of uniform arbitrary cross-section, Kelland obtained the equation,
$$U^2 = gA/b,$$
for U, the wave velocity of long waves moving along the canal.

Kelland, however, was successful in obtaining an exact expression for the motion of progressive waves in a straight canal the sides of which sloped down uniformly to an edge at an angle of 45°; he found that, taking the axis of x along this edge, we can put

$$\phi = A \cosh my \cosh mz \cos \sqrt{2}\,(mx - nt),$$

satisfying the equation of continuity

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0,$$

and also the boundary condition $\frac{\partial\phi}{\partial\nu} = 0$;

or,
$$l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z} = 0,$$

l, m, n denoting the direction cosines of the normal to the boundary. In this case the boundary conditions are

$$\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial z} = 0, \text{ when } y - z = 0,$$

$$\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z} = 0, \text{ when } y + z = 0,$$

which are immediately seen to be satisfied.

At the surface of the water, where $z = h$,

$$l\frac{\partial\phi}{\partial z} = \phi,$$

for all values of x and y; or,

$$ml \sinh mh = \cosh mh, \text{ or } ml = \coth mh,$$

and

$$U^2 = \frac{n^2}{m^2} = \frac{g}{lm^2} = \frac{g}{m}\tanh mh,$$

the same as for waves in water of depth h, but now $2\pi/\lambda = m\sqrt{2}$.

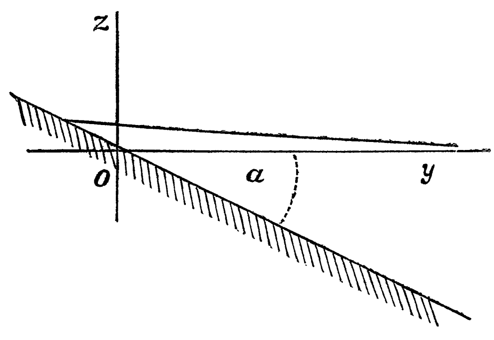
By transferring the axis of x to the edge of the water on one bank, we obtain $\phi = A \cosh m(y + h) \cosh m(z - h) \cos \sqrt{2}(mx - nt)$, which, when h is made indefinitely great, can be replaced by

$$\phi = Be^{-m(y-z)} \cos \sqrt{2}(mx - nt),$$

giving the motion for waves moving parallel to a shore sloping at 45°, the crests of the waves being perpendicular to the shore.

This may easily be generalized, as Stokes has shown (Report on Recent Researches in Hydrodynamics, Math. and Phys. Papers, I, p. 167), for a shore sloping at any angle α, by putting

$$\phi = Be^{-m(y\cos\alpha - z\sin\alpha)} \cos(mx - nt),$$



satisfying the equation of continuity and the boundary condition

$$\frac{\partial\phi}{\partial y}\sin\alpha + \frac{\partial\phi}{\partial z}\cos\alpha = 0, \text{ when } y\sin\alpha + z\cos\alpha = 0;$$

also, at the free surface $z=0$,

$$l\frac{\partial\phi}{\partial z}=\phi;\ \text{or, } ml\sin\alpha=1;$$

so that
$$U^2=\frac{n^2}{m^2}=\frac{g}{lm^2}=\frac{g}{m}\sin\alpha=\frac{g\lambda}{2\pi}\sin\alpha,$$

α denoting here the slope of the shore to the horizon.

Analogous to Kelland's previous solution, we might put

$$\phi=A\sinh my\sinh mz\sin\sqrt{2}\,(mx-nt),$$

satisfying the equation of continuity and the boundary conditions, and at the free surface $z=h$, $l\dfrac{\partial\phi}{\partial z}=\phi$ gives

$$ml=\tanh mh;$$

or,
$$U^2=\frac{g}{m}\coth mh.$$

The shape of the free surface will be different in the two cases; in the first, Kelland's case, when $z=h$,

$$\frac{d\eta}{dt}=\frac{\partial\phi}{\partial z}=mA\sinh mh\cosh my\cos\sqrt{2}\,(mx-nt);$$

so that
$$\eta=-\frac{mA}{n\sqrt{2}}\sinh mh\cosh my\sin\sqrt{2}\,(mx-nt)$$

or the form $\eta=a\cosh my\sin\sqrt{2}\,(mx-nt)$,

an even function of y; and in the second case,

$$\frac{d\eta}{dt}=\frac{\partial\phi}{\partial z}=mA\cosh mh\sinh my\sin\sqrt{2}\,(mx-nt);$$

so that the free surface is of the form

$$\eta=a\sinh my\cos\sqrt{2}\,(mx-nt),$$

an odd function of y.

Introducing capillarity on the free surface, but neglecting its effect at the contact of the surface with the bank, then the equation

$$l\frac{\partial\phi}{\partial z}+\frac{Tl}{g\rho}\frac{\partial^3\phi}{\partial z^3}=\phi$$

gives
$$ml\sinh mh+\frac{Tlm^3}{g\rho}\sinh mh=\cosh mh;$$

or,
$$ml+\frac{Tlm^3}{g\rho}=\coth mh;$$

and, for the second kind of motion,

$$ml+\frac{Tlm^3}{g\rho}=\tanh mh.$$

17. Let us apply Kelland's expression to determine the progressive waves at the surface of separation of two liquids, each half filling a pipe, of which the cross-section is a square with a vertical diagonal, of length $2h$.

Taking diagonals of the square as axes of y and z, then we can put

$$\phi = A \cosh my \cosh m(z+h) \cos \sqrt{2}(mx-nt),$$
$$\phi' = -A \cosh my \cosh m(z-h) \cos \sqrt{2}(mx-nt),$$

satisfying the equation of continuity and the boundary conditions, except just where the surface of separation meets the boundary, the disturbing effect of which we shall neglect, although of course disturbing waves would be generated thereby.

Then, as before, if η denotes the elevation of the surface of separation where the mean value of $z = 0$,

$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} = mA \cosh my \sinh mh \cos \sqrt{2}(mx-nt);$$

so that

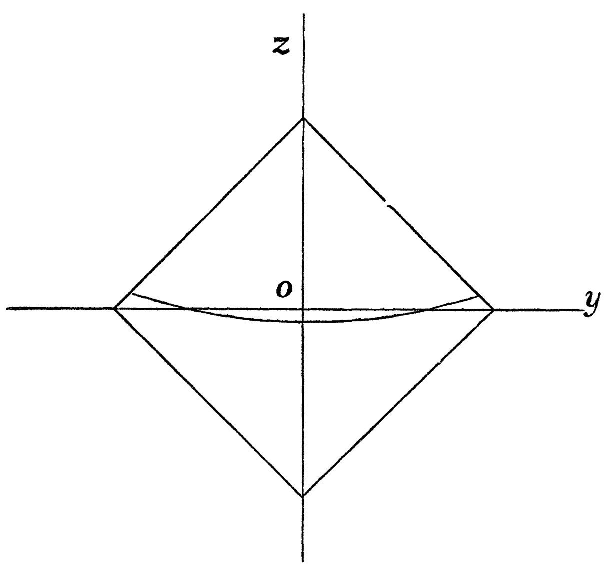
$$\eta = -\frac{mA}{n\sqrt{2}} \cosh my \sinh mh \sin \sqrt{2}(mx-nt);$$

also,

$$\frac{d\phi}{dt} = nA\sqrt{2} \cosh my \cosh mh \sin \sqrt{2}(mx-nt) = -2\frac{n^2}{m}\eta \coth mh,$$

and

$$\frac{d\phi'}{dt} = 2\frac{n^2}{m}\eta \coth mh.$$



Also, as before,

$$\partial p + g\rho\eta + \rho\frac{d\phi}{dt} = 0,$$

$$\partial p' + g\rho'\eta + \rho'\frac{d\phi'}{dt} = 0;$$

or,

$$\partial p + g\rho\eta - 2\rho\frac{n^2}{m}\eta \coth mh = 0,$$

$$\partial p' + g\rho'\eta + 2\rho'\frac{n^2}{m}\eta \coth mh = 0;$$

so that if $\partial p = \partial p'$,

$$g(\rho - \rho') = 2\frac{n^2}{m}(\rho + \rho')\coth mh;$$

or,

$$U^2 = \frac{n^2}{m^2} = \frac{1}{2}\frac{g}{m}\frac{\rho - \rho'}{\rho + \rho'}\tanh mh.$$

Similarly, by putting

$$\phi = A \sinh my \sinh m(z + h) \sin \sqrt{2}(mx - nt),$$

$$\phi' = A \sinh my \sinh m(z - h) \sin \sqrt{2}(mx - nt),$$

we should obtain

$$U^2 = \frac{1}{2}\frac{g}{m}\frac{\rho - \rho'}{\rho + \rho'}\coth mh.$$

18. *Standing Waves across a Rectangular Channel.*

We shall find that, by replacing the hyperbolic functions of y and z partly by circular functions in the above solutions for progressive waves along the channel, we shall be able to solve the motion of standing waves in which the crests are parallel to the axis of the canal or channel. For, if we put

$$\phi = A(\cos my \cosh mz + \cosh my \cos mz)\cos nt,$$

or

$$\phi = A(\sin my \sinh mz + \sinh my \sin mz)\sin nt,$$

expressions which are equivalent to those obtained by Kirchoff (Ueber stehende Schwingungen einer schweren Flüssigkeit, Gesammelte Abhandlungen, II, p. 440), we shall satisfy the equation of continuity and the boundary conditions

$$\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial z} = 0, \text{ when } y - z = 0,$$

$$\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z} = 0, \text{ when } y + z = 0;$$

and at the free surface $z = h$ the condition

$$l\frac{\partial\phi}{\partial z} = \phi,$$

for all values of y, leads to the equations

$$ml(\cos my \sinh mh - \cosh my \sin mh) = \cos my \cosh mh + \cosh my \cos mh,$$

giving $\quad ml \sinh mh = \cosh mh$ and $-ml \sin mh = \cos mh,$

or $$ml = \coth mh = -\cot mh,$$

for the even vibrations, and

$$ml(\sin my \cosh mh + \sinh my \cos mh) = \sin my \sinh mh + \sinh my \sin mh,$$

giving $\quad ml \cosh mh = \sinh mh$ and $ml \cos mh = \sin mh,$

or $$ml = \tanh mh = \tan mh,$$

for the odd vibrations.

In these equations mh is the same as Kirchoff's p, and, with the notation of the hyperbolic functions, Kirchoff's period equations

$$\sinh p = \frac{\sin p}{\sqrt{(\cos 2p)}}$$

and $$\cosh p = \frac{\cos p}{\sqrt{(\cos 2p)}}$$

correspond to our period equations

$$\coth p = -\cot p$$

and $$\tanh p = \tan p;$$

both being included in the single equation

$$\cos 2p \cosh 2p = 1,$$

the period equation for the lateral vibrations of a free-free or clamped-clamped bar (Rayleigh, Sound, I, p. 219).

In fact, the vibrations of the surface are of exactly the same character as those of a free-free bar of length $2h$, the first value of ϕ giving the even and the second the odd vibrations.

Suppose the surface were covered with ice of thickness e and flexural rigidity $L = \frac{1}{12} e^3 E$; then, at the surface $z = h$,

$$\rho e \frac{d^2\eta}{dt^2} = -L \frac{d^4\eta}{dx^4} + \partial p$$

and $$\partial p + g\rho\eta + \rho \frac{\partial \phi}{\partial t} = 0.$$

But $$\frac{d^2\eta}{dt^2} = -n^2\eta, \quad \frac{d^4\eta}{dx^4} = m^4\eta;$$

so that $$n^2\rho e\eta = m^4 L\eta + g\rho\eta + \rho \frac{\partial \phi}{\partial t},$$

or
$$\{\rho(g - n^2 e) + m^4 L\}\,\eta = -\rho\frac{\partial\phi}{\partial t};$$
and since
$$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z},\ \frac{\partial^2\phi}{\partial t^2} = -n^2\phi,$$
$$\{\rho(g - n^2 e) + m^4 L\}\frac{\partial\phi}{\partial z} = n^2\rho\phi;$$
or since $g = ln^2$,
$$\left(l - e + \frac{m^4 L}{n^2\rho}\right)\frac{\partial\phi}{\partial z} = \phi,$$
and, therefore, as before,
$$m(l-e) + \frac{m^5 L}{n^2\rho} = \coth mh = -\cot mh,$$
or
$$= \tanh mh = \tan mh,$$
showing that the length of the equivalent simple pendulum is altered by $m^4.L/n^2\rho - e$ by the presence of the ice.

Dropping for the present the factors $A\cos nt$ and $A\sin nt$, then
$$\phi = \cos my\cosh mz + \cosh my\cos mz,$$
or
$$\phi = \sin my\sinh mz + \sinh my\sin mz,$$
and, therefore, the conjugate current functions are
$$\psi = \sin my\sinh mz - \sinh my\sin mz,$$
or
$$\psi = -\cos my\cosh mz + \cosh my\cos mz;$$
so that
$$\phi + i\psi = \cos m(z+iy) + \cosh m(z+iy),$$
or
$$= i\cos m(z+iy) - i\cosh m(z+iy).$$

Denoting $\phi + i\psi$ by w, and $z + iy$ by u, then
$$w = \cos mu + \cos imu,$$
or
$$w = i\cos mu - i\cos imu,$$
gives the required motion in a rectangular channel.

By transferring the axis of x to the edge of the water on one bank, we obtain $\phi = \cos m(y-h)\cosh m(z+h) + \cosh m(y-h)\cos m(z+h)$,
or $\phi = \sin m(y-h)\sinh m(z+h) + \sinh m(y-h)\sin m(z+h)$;
and these are the co-ordinates employed by Kirchoff.

When h is made indefinitely great, these expressions may be replaced by (omitting constant factors, and remembering that the axis of z is drawn vertically *upwards*),
$$\phi = e^{mz}(\cos my - \sin my) + e^{-my}(\cos mz - \sin mz),$$
and then, when $z = 0$,
$$l\frac{\partial\phi}{\partial z} = \phi,$$
if
$$ml = 1,$$

giving Kirchoff's solution of standing waves parallel to a shore sloping at 45° (Abhandlungen, II, p. 434).

Returning to the original axes, the second value of ϕ gives, when m is small, so that m^2, ... may be neglected,

$$\phi = 2myz;$$

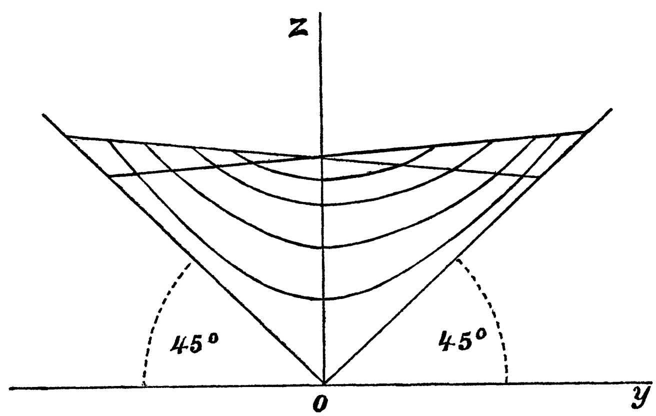
or, restoring the periodic factor, we may put

$$\phi = 2myz \sin nt,$$

and then $$l = h;$$

also, $$\frac{d\eta}{dt} = \frac{\partial\phi}{\partial z} = 2my \sin nt;$$

so that $$\eta = -2\frac{m}{n} y \cos nt,$$



so that the free surface of the liquid remains plane during this kind of wave motion. Also, $$\psi = m(y^2 - z^2) \sin nt,$$

so that the liquid particles oscillate in rectangular hyperbolas (Kirchoff, Abhandlungen, II, p. 436).

19. *Standing Waves across a Channel of* 120°.

Let us now attempt the solution of the corresponding waves in a canal the sides of which slope at angles of 30° to the horizon, and are therefore inclined to each other at an angle of 120°.

First, we notice that we can begin with an algebraical solution by putting, as the simplest case,

$$\phi = A\Phi \cos nt,$$

where $$\Phi = z^3 - 3y^2z + h^3,$$

and the corresponding stream function

$$\Psi = 3yz^2 - y^3$$
$$= y(z\sqrt{3} - y)(z\sqrt{3} + y),$$

which vanishes when $y = \pm z\sqrt{3}$, showing that the boundary conditions are satisfied, and also,

$$\Phi + i\Psi = (z + iy)^3 + h^3.$$

Then, at the free surface $z = h$,

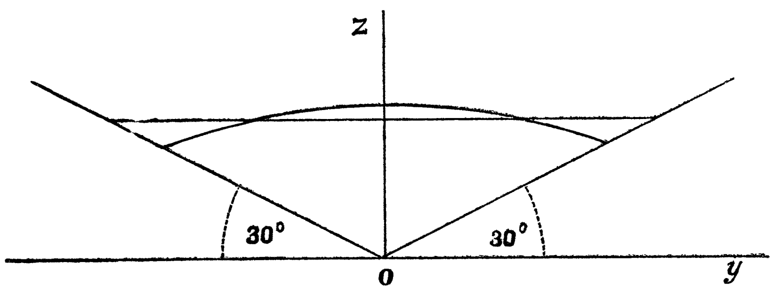
$$\frac{\partial\Phi}{\partial z} = 3h^2 - 3y^2, \quad \Phi = 2h^3 - 3hy^2;$$

so that $$h\frac{\partial\Phi}{\partial z} = \Phi,$$

and, therefore, $l = h$.

The free surface is now a parabolic cylinder, for

$$\frac{d\eta}{dt} = 3A(h^2 - y^2)\cos nt,$$
$$\eta = \frac{3A}{n}(h^2 - y^2)\sin nt.$$



For waves of a higher order, let us attempt the solution by putting

$$w = \cos mu + \cos m\beta u + \cos m\beta^2 u,$$

where $$u = z + iy,$$
$$w = \Phi + i\Psi,$$

and $$\beta^3 = -1, \ \beta = \frac{1}{2} - \frac{1}{2}i\sqrt{3}, \ \beta^2 = -\frac{1}{2} - \frac{1}{2}i\sqrt{3}.$$

Then, at the boundary $y = z\sqrt{3}$,

$$w = \cos(1 + i\sqrt{3})mz + \cos\frac{1}{2}(1 - i\sqrt{3})(1 + i\sqrt{3})mz + \cos\frac{1}{2}(1 + i\sqrt{3})^2 mz$$

$$= \cos(1 + i\sqrt{3})mz + \cos 2mz + \cos(-1 + i\sqrt{3})z,$$

a real quantity, so that $\Psi = 0$.

Again, at the boundary $y = -z\sqrt{3}$, w is real and $\psi = 0$. These conditions will also be satisfied by putting

$$w = \sin mu - \sin m\beta u + \sin m\beta^2 u,$$

so that generally we can put

$$w = \sin m(u - \alpha) - \sin m(\beta u + \alpha) + \sin m(\beta^2 u - \alpha);$$

so that, since

$$\beta u + \alpha = \frac{1}{2}(y\sqrt{3} + z + 2\alpha) + \frac{1}{2}i(y - z\sqrt{3}),$$

$$\beta^2 u - \alpha = \frac{1}{2}(y\sqrt{3} - z - 2\alpha) - \frac{1}{2}i(y + z\sqrt{3}),$$

$$\Phi = \sin m(z - \alpha)\cosh my - \sin\frac{1}{2}m(y\sqrt{3} + z + 2\alpha)\cosh\frac{1}{2}m(y - z\sqrt{3})$$
$$+ \sin\frac{1}{2}m(y\sqrt{3} - z - 2\alpha)\cosh\frac{1}{2}m(y + z\sqrt{3}),$$

$$\Psi = \cos m(z - \alpha)\sinh my - \cos\frac{1}{2}m(y\sqrt{3} + z + 2\alpha)\sinh\frac{1}{2}m(y - z\sqrt{3})$$
$$- \cos\frac{1}{2}m(y\sqrt{3} - z - 2\alpha)\sinh\frac{1}{2}m(y + z\sqrt{3}).$$

Putting $y = z\sqrt{3}$,

$$\Psi = \cos m(z - \alpha)\sinh mz\sqrt{3} - \cos m(z - \alpha)\sinh mz\sqrt{3} = 0,$$

and putting $y = -z\sqrt{3}$,

$$\Psi = -\cos m(z - \alpha)\sinh mz\sqrt{3} - \cos m(-z + \alpha)\sinh(-mz\sqrt{3}) = 0,$$

so that the boundary conditions are satisfied.

Expanded in ascending powers of $(z + ix)$, we shall find

$$w = -3\sin\alpha - \frac{1}{2}m^3(z + ix)^3\cos\alpha + \ldots.,$$

so that when m is small, and $m^5, \ldots.$ can be neglected, we obtain the previous algebraical solution.

At the free surface $z = h$ we must have

$$l\frac{\partial\Phi}{\partial z} = \Phi$$

for all values of y; and, therefore, since we may write

$$\Phi = \sin m(z-\alpha)\cosh my$$
$$-2\sin\frac{1}{2}m(z+2\alpha)\cosh\frac{1}{2}mz\surd 3\cosh\frac{1}{2}my\cos\frac{1}{2}my\surd 3$$
$$+2\cos\frac{1}{2}m(z+2\alpha)\sinh\frac{1}{2}mz\surd 3\sinh\frac{1}{2}my\sin\frac{1}{2}my\surd 3,$$

$$l\frac{\partial\Phi}{\partial z} = ml\cos m(z-\alpha)\cosh my$$
$$-ml\left\{\cos\frac{1}{2}m(z+2\alpha)\cosh\frac{1}{2}mz\surd 3+\surd 3\sin\frac{1}{2}m(z+2\alpha)\right.$$
$$\left.\sinh\frac{1}{2}mz\surd 3\right\}\cosh\frac{1}{2}my\cos\frac{1}{2}my\surd 3$$
$$+ml\left\{-\sin\frac{1}{2}m(z+2\alpha)\sinh\frac{1}{2}mz\surd 3+\surd 3\cos\frac{1}{2}m(z+2\alpha)\right.$$
$$\left.\cosh\frac{1}{2}mz\surd 3\right\}\sinh\frac{1}{2}my\sin\frac{1}{2}mz\surd 3;$$

therefore, at the free surface $z=h$,

$$ml\cos m(h-\alpha) = \sin m(h-\alpha), \tag{I}$$

$$ml\left\{\cos\frac{1}{2}m(h+2\alpha)\cosh\frac{1}{2}mh\surd 3+\surd 3\sin\frac{1}{2}m(h+2\alpha)\sinh\frac{1}{2}mh\surd 3\right\}$$
$$=2\sin\frac{1}{2}m(h+2\alpha)\cosh\frac{1}{2}mh\surd 3, \tag{II}$$

$$ml\left\{-\sin\frac{1}{2}m(h+2\alpha)\sinh\frac{1}{2}mh\surd 3+\surd 3\cos\frac{1}{2}m(h+2\alpha)\cosh\frac{1}{2}mh\surd 3\right\}$$
$$=2\cos\frac{1}{2}m(h+2\alpha)\sinh\frac{1}{2}mh\surd 3, \tag{III}$$

three equations for determining mh, α and l.

From (II) and (III), eliminating ml,

$$\cos^2\frac{1}{2}m(h+2\alpha)\sinh mh\surd 3+\surd 3\sin m(h+2\alpha)\sinh^2\frac{1}{2}mh\surd 3$$
$$=-\sin^2\frac{1}{2}m(h+2\alpha)\sinh mh\surd 3+\surd 3\sin m(h+2\alpha)\cosh^2\frac{1}{2}mh\surd 3;$$

or,
$$\sinh mh\surd 3 = \surd 3\sin m(h+2\alpha). \tag{IV}$$

Also, from (II) and (III),

$$\tan\frac{1}{2}m(h+2\alpha) = \frac{ml\cosh\frac{1}{2}mh\surd 3}{2\cosh\frac{1}{2}mh\surd 3 - ml\surd 3\sinh\frac{1}{2}mh\surd 3}$$
$$=\frac{ml\surd 3\cosh\frac{1}{2}mh\surd 3 - 2\sinh\frac{1}{2}mh\surd 3}{ml\sinh\frac{1}{2}mh\surd 3};$$

so that

$$\frac{1}{2} m^2l^2 \sinh mh\surd 3 = 2ml\surd 3 \left(\cosh^2 \frac{1}{2} mh\surd 3 + \sinh^2 \frac{1}{2} mh\surd 3\right)$$
$$- \frac{3}{2} m^2l^2 \sinh mh\surd 3 - 2 \sinh mh\surd 3;$$

or,
$$ml + \frac{1}{ml} = \surd 3 \coth mh\surd 3. \qquad \text{(V)}$$

From (I), $ml = \tan m(h - \alpha)$,
so that

$$ml + \frac{1}{ml} = 2 \operatorname{cosec} 2m(h - \alpha);$$

or,
$$\sin 2m(h - \alpha) = \frac{2}{\surd 3} \tanh mh\surd 3. \qquad \text{(VI)}$$

From (IV) and (VI),

$$\sin mh \cos 2m\alpha + \cos mh \sin 2m\alpha = \frac{1}{\surd 3} \sinh mh\surd 3,$$
$$\sin 2mh \cos 2m\alpha - \cos 2mh \sin 2m\alpha = \frac{2}{\surd 3} \tanh mh\surd 3;$$

and, therefore,

$$\surd 3 \sin 3mh \cos 2m\alpha = \sinh mh\surd 3 \left(\cos 2mh + \frac{2\cos mh}{\cosh mh\surd 3}\right),$$
$$\surd 3 \sin 3mh \sin 2m\alpha = \sinh mh\surd 3 \left(\sin 2mh - \frac{2\sin mh}{\cosh mh\surd 3}\right).$$

Squaring and adding, to eliminate α,

$$\frac{3\sin^2 3mh}{\sinh^2 mh\surd 3} = 1 + \frac{4\cos 3mh}{\cosh mh\surd 3} + \frac{4}{\cosh^2 mh\surd 3};$$

or, denoting $\cos 3mh$ by α, and $\cosh mh\surd 3$ by β,

$$3\frac{1-\alpha^2}{\beta^2-1} = 1 + 4\frac{\alpha}{\beta} + \frac{4}{\beta^2},$$

reducing to

$$\beta^4 + 4\alpha\beta^3 + 3\alpha^2\beta^2 - 4\alpha\beta - 4 = 0,$$

or
$$(\beta^2 + 2\alpha\beta)^2 = (\alpha\beta + 2)^2,$$
$$\beta^2 + 2\alpha\beta = \pm(\alpha\beta + 2);$$

so that
$$\beta^2 + \alpha\beta = 2,$$

or
$$\beta^2 + 3\alpha\beta = -2;$$
$$\alpha = -\beta + 2/\beta,$$

or
$$3\alpha = -\beta - 2/\beta;$$
$$\cos 3mh = -\cosh mh\surd 3 + 2\operatorname{sech} mh\surd 3,$$

or
$$3\cos 3mh = -\cosh mh\surd 3 - 2\operatorname{sech} mh\surd 3,$$

the period equations.